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Convolution operators on the weighted Herz-type Hardy spaces

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Abstract

A molecular characterization of the weighted Herz-type Hardy spaces $H\dot{K}_q^{n(1/p-1/q),p}(w,w)$ and $HK_q^{n(1/p-1/q),p}(w,w)$ is given, by which the boundedness of the Hilbert transform and the Riesz transforms are proved on these space for $0<p\leqslant 1$. These results are obtained by first deriving that the convolution operator $Tf=k*f$ is bounded on the weighted Herz-type Hardy spaces.

MSC: 42B20; 42B30

Keywords: Central atom; Molecular characterization; Weighted Herz-type Hardy space

1. Introduction

In 1964, Beurling [2] first introduced some fundamental form of Herz spaces to study convolution algebras. Four years later Herz [7] gave versions of the spaces defined below in a slightly different setting. Since then, the theory of Herz spaces has been significantly developed, and these spaces have turned out to be quite useful in analysis. For example, they were used by Baernstein and Sawyer [1] to characterize the multipliers on the standard Hardy spaces, and used by Lu and Yang [17] in the study on partial differential equations.

On the other hand, a theory of Hardy spaces associated with Herz spaces has been developed for more than a decade (see [4,15]). These new Hardy spaces can be regarded as the local version at the origin of the classical Hardy spaces H^p and are good substitutes for H^p when we study the boundedness of non-translation invariant operators (see [16]). For the weighted case, Lu and

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Yang [13,14] introduced the following weighted Herz-type Hardy spaces and established their atomic decompositions.

Let $Q_k = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| \leqslant 2^k, i = 1, \ldots, n\right\}$ and $C_k = Q_k \backslash Q_{k-1}$ for $k \in \mathbb{Z}$, χ_k be the characteristic function of the set C_k. For a non-negative weight function w, we set $w(E) = \int_E w(x)\,dx$ and write $L^q_w(\mathbb{R}^n)$ to be the set of all functions f satisfying

$$\|f\|_{L^q_w} = \left(\int_{\mathbb{R}^n} |f(x)|^q w(x)\,dx\right)^{1/q} < \infty.$$

In what follows, if $w \equiv 1$, we will denote $L^q_w(\mathbb{R}^n)$ by $L^q(\mathbb{R}^n)$.

Definition. Let $\alpha \in \mathbb{R}$, $0 < p, q < \infty$, and w_1 and w_2 be non-negative weight functions.

(a) The *homogeneous weighted Herz space* $\dot{K}^{\alpha,p}_q(w_1; w_2)$ is the set of all functions $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\}, w_2(x)\,dx)$ satisfying $\|f\|_{\dot{K}^{\alpha,p}_q(w_1;w_2)} < \infty$, where

$$\|f\|^p_{\dot{K}^{\alpha,p}_q(w_1;w_2)} = \sum_{k\in\mathbb{Z}} [w_1(Q_k)]^{\alpha p/n} \|f\chi_k\|^p_{L^q_{w_2}}.$$

(b) The *non-homogeneous weighted Herz space* $K^{\alpha,p}_q(w_1; w_2)$ is the set of all functions $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n, w_2(x)\,dx)$ satisfying $\|f\|_{K^{\alpha,p}_q(w_1;w_2)} < \infty$, where

$$\|f\|^p_{K^{\alpha,p}_q(w_1;w_2)} = [w_1(Q_0)]^{\alpha p/n} \|f\chi_{Q_0}\|^p_{L^q_{w_2}} + \sum_{k=1}^{\infty} [w_1(Q_k)]^{\alpha p/n} \|f\chi_k\|^p_{L^q_{w_2}}.$$

Let $Gf(x)$ be the grand maximal function of $f(x)$ defined by

$$Gf(x) = \sup_{\phi\in A_N} \sup_{\substack{|x-y|<t\\ t>0}} |(f * \phi_t)(y)|,$$

where $A_N = \left\{\phi \in \mathcal{S}(\mathbb{R}^n) : \sup_{|\gamma|,|\beta|\leqslant N} |x^\gamma D^\beta \phi(x)| \leqslant 1\right\}$ for N sufficiently large.

Definition. Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $1 < q < \infty$ and w_1 and w_2 be non-negative weight functions.

(a) The *homogeneous weighted Herz-type Hardy space* $H\dot{K}^{\alpha,p}_q(w_1; w_2)$ associated with the space $\dot{K}^{\alpha,p}_q(w_1; w_2)$ is defined by

$$H\dot{K}^{\alpha,p}_q(w_1; w_2) = \left\{f \in S'(\mathbb{R}^n) : Gf \in \dot{K}^{\alpha,p}_q(w_1; w_2)\right\}.$$

with norm $\|f\|_{H\dot{K}^{\alpha,p}_q(w_1;w_2)} = \|Gf\|_{\dot{K}^{\alpha,p}_q(w_1;w_2)}$.

(b) The *non-homogeneous weighted Herz-type Hardy space* $HK^{\alpha,p}_q(w_1; w_2)$ associated with the space $K^{\alpha,p}_q(w_1; w_2)$ is defined by

$$HK^{\alpha,p}_q(w_1; w_2) = \left\{f \in S'(\mathbb{R}^n) : Gf \in K^{\alpha,p}_q(w_1; w_2)\right\}.$$

with norm $\|f\|_{HK^{\alpha,p}_q(w_1;w_2)} = \|Gf\|_{K^{\alpha,p}_q(w_1;w_2)}$.

Throughout this paper C denotes a constant not necessarily the same at each occurrence, and a subscript is added when we wish to make clear its dependence on the parameter in the subscript. We also use $a \approx b$ to denote the equivalence of a and b; that is, there exist two positive constants C_1, C_2 independent of a, b such that $C_1 a \leqslant b \leqslant C_2 a$.

2. The A_1 weights

The definition of weighted class A_p was first used by Muckenhoupt [18], Coifman–Fefferman [3], and Hunt–Muckenhoupt–Wheeden [8] in the investigation of weighted L^p boundedness of Hardy–Littlewood maximal function and Hilbert transform. In this article a weight means the A_1 weight. More precisely, let w be a nonnegative function defined on $\mathbb{R}^n$. We say that $w \in A_1$ if

$$\frac{1}{|I|}\int_I w(x)\,dx \leqslant C \cdot \operatorname*{ess\,inf}_{x\in I} w(x) \quad \text{for every cube } I \subseteq \mathbb{R}^n.$$

For any cube I and $\lambda > 0$, we shall denote by λI the cube concentric with I which is λ times as long. It is known that for $w \in A_1$, w satisfies the *doubling condition*; that is, there exists an absolute constant C such that $w(2I) \leqslant C w(I)$. A more specific estimate for $w(\lambda I)$ is given as follows.

Theorem A (*Garcia-Cuerva and Rubio de Francia [5]*). *Let $w \in A_1$. Then, for any cube I and $\lambda > 1$,*

$$w(\lambda I) \leqslant C\lambda^n w(I),$$

where C is independent of I and λ.

If there exist $r > 1$ and a fixed constant $C > 0$ such that

$$\left(\frac{1}{|I|}\int_I w^r(x)\,dx\right)^{1/r} \leqslant \frac{C}{|I|}\int_I w(x)\,dx \quad \text{for all cubes } I \subseteq \mathbb{R}^n,$$

we say that w satisfies *reverse Hölder condition* and write $w \in RH_r$. It follows from Hölder's inequality that $w \in RH_r$ implies $w \in RH_s$ for $s < r$. It is known that if $w \in RH_r$, $r > 1$, then $w \in RH_{r+\varepsilon}$ for some $\varepsilon > 0$. We thus write $r_w \equiv \sup\{r > 1 : w \in RH_r\}$ to denote the *critical index of w for the reverse Hölder condition*.

The following result provides us with the comparison between the Lebesgue measure of a set E and its weighted measure $w(E)$.

Theorem B (*Garcia-Cuerva and Rubio de Francia [5], Gundy and Wheeden [6]*). *Let $w \in A_1 \cap RH_r$, $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1\frac{|E|}{|I|} \leqslant \frac{w(E)}{w(I)} \leqslant C_2\left(\frac{|E|}{|I|}\right)^{(r-1)/r}$$

for any measurable subset E of a cube I.

3. The atomic decomposition and molecular characterization

Lu and Yang [13,14] gave the following definition of atoms in the weighted Herz-type Hardy space and its atomic decomposition.

Definition. Let $w_1, w_2 \in A_1$, $1 < q < \infty$, $n(1-1/q) \leqslant \alpha < \infty$, and s be a non-negative integer greater than or equal to $N \equiv [\alpha + n(1/q - 1)]$, where $[\,\cdot\,]$ is the integer function.

(a) A function a on $\mathbb{R}^n$ is called a *central* (α, q, s)*-atom with respect to* (w_1, w_2) (or a *central* $(\alpha, q, s; w_1, w_2)$*-atom*), if it satisfies
 (i) $\operatorname{supp} a \subseteq B(0, R)$, $R > 0$,
 (ii) $\|a\|_{L^q_{w_2}} \leqslant w_1(B(0,R))^{-\alpha/n}$,
 (iii) $\int_{\mathbb{R}^n} a(x) x^\beta \, dx = 0$ for every multi-index β with $|\beta| \leqslant s$.

(b) A function a on $\mathbb{R}^n$ is called a *central* (α, q, s)*-atom of restricted type with respect to* (w_1, w_2) (or a *central* $(\alpha, q, s; w_1, w_2)$*-atom of restricted type*), if it satisfies (ii), (iii), and
 (i′) $\operatorname{supp} a \subseteq B(0, R)$, $R \geqslant 1$.

Theorem C. *Let* $w_1, w_2 \in A_1$, $0 < p \leqslant 1 < q < \infty$, *and* $n(1 - 1/q) \leqslant \alpha < \infty$. *Then* $f \in H\dot{K}_q^{\alpha,p}(w_1; w_2)$ (*or* $HK_q^{\alpha,p}(w_1; w_2)$) *if and only if*

$$f(x) \overset{S'}{=} \sum_{k=-\infty}^{\infty} \lambda_k a_k(x) \quad \left(or \quad f(x) \overset{S'}{=} \sum_{k=0}^{\infty} \lambda_k a_k(x) \right),$$

where each a_k *is a central* $(\alpha, q, s; w_1, w_2)$*-atom* (*or for non-homogeneous spaces,* a_k *is a central* $(\alpha, q, s; w_1, w_2)$*-atom of restricted type*), *and* $\sum_{k=-\infty}^{\infty} |\lambda_k|^p < \infty$ $\left(or\ \sum_{k=0}^{\infty} |\lambda_k|^p < \infty\right)$. *Moreover,*

$$\|f\|_{H\dot{K}_q^{\alpha,p}(w_1;w_2)} \approx \inf \left(\sum_{k=-\infty}^{\infty} |\lambda_k|^p \right)^{1/p}$$

$$\left(or \quad \|f\|_{HK_q^{\alpha,p}(w_1;w_2)} \approx \inf \left(\sum_{k=0}^{\infty} |\lambda_k|^p \right)^{1/p} \right),$$

where the infimum is taken over all the above decompositions of f (*the symbol* "$\overset{S'}{=}$" *denotes convergence in the sense of tempered distributions*).

We recall the w-(p, q, s)-atom in the classical weighted Hardy spaces (cf. [10]). Let $0 < p \leqslant 1 < q \leqslant \infty$ and $w \in A_1$. For $s \in \mathbb{Z}$ satisfying $s \geqslant [n(1/p - 1)]$, a real-valued function $a(x)$ is called w-(p, q, s)*-atom centered at* 0 if

(i) $a \in L^q_w(\mathbb{R}^n)$ and is supported in a cube I with center 0,
(ii) $\|a\|_{L^q_w} \leqslant w(I)^{1/q - 1/p}$,
(iii) $\int_{\mathbb{R}^n} a(x) x^\alpha \, dx = 0$ for every multi-index α with $|\alpha| \leqslant s$.

In the definition of central atom, if we set $\alpha = n(1/p - 1/q)$ and consider $w = w_1 = w_2 \in A_1$, then $N = [n(1/p - 1)]$. Thus, for $1 < q < \infty$, the w-(p, q, s)-atom centered at 0 in the classical weighted Hardy spaces coincides with the central $(n(1/p-1/q), q, s; w, w)$-atom. Denote by I_r the cube centered at the origin with side length $2r$. The following definition of molecule established by Lee and Lin [10] is meaningful in the weighted Herz-type Hardy spaces.

Definition. For $0 < p \leqslant 1 < q < \infty$, let $w \in A_1$ with critical index r_w for the reverse Hölder condition. Set $s \geqslant [n(1/p-1)]$, $\varepsilon > \max\{\frac{sr_w}{n(r_w-1)} + \frac{1}{(r_w-1)}, \frac{1}{p} - 1\}$, $a = 1 - 1/p + \varepsilon$, and $b = 1 - 1/q + \varepsilon$.

(a) A *central* (p, q, s, ε)*-molecule with respect to* w (or a *central* w-(p, q, s, ε)*-molecule*) is a function $M \in L^q_w(\mathbb{R}^n)$ satisfying
  (i) $M(x) \cdot w(I_{|x|})^b \in L^q_w(\mathbb{R}^n)$,
  (ii) $\|M\|_{L^q_w}^{a/b} \cdot \left\|M(x) \cdot w(I_{|x|})^b\right\|_{L^q_w}^{1-a/b} \equiv \mathfrak{N}_w(M) < \infty$,
  (iii) $\int_{\mathbb{R}^n} M(x) x^\alpha \, dx = 0$ for every multi-index α with $|\alpha| \leqslant s$.
(b) A function $M \in L^q_w(\mathbb{R}^n)$ is called a *central* (p, q, s, ε)*-molecule of restricted type with respect to* w (or a *central* w-(p, q, s, ε)*-molecule of restricted type*) if it satisfies (i)–(iii), and
  (iv) $\|M\|_{L^q_w} \leqslant w(I_1)^{1/q-1/p}$.

The above $\mathfrak{N}_w(M)$ is called the molecular norm of M with respect to w (or w-molecular norm of M). If there is no ambiguity, we still use $\mathfrak{N}(M)$ to denote the w-molecular norm of M.

Following from [10, Theorem 1], we immediately have the molecular characterization of the weighted Herz-type Hardy spaces.

Theorem 1. *Let* (p, q, s, ε) *be the quadruple in the definition of central molecule, and let* $w \in A_1$.

(a) *Every central* (p, q, s, ε)*-molecule* M *with respect to* w *belongs to* $H\dot{K}_q^{n(1/p-1/q),p}(w; w)$ *and* $\|M\|_{H\dot{K}_q^{n(1/p-1/q),p}(w;w)} \leqslant C\mathfrak{N}(M)$, *where the constant* C *is independent of* M.
(b) *Every central* (p, q, s, ε)*-molecule of restricted type* M *with respect to* w *belongs to* $HK_q^{n(1/p-1/q),p}(w; w)$ *and* $\|M\|_{HK_q^{n(1/p-1/q),p}(w;w)} \leqslant C\mathfrak{N}(M)$, *where the constant* C *is independent of* M.

Proof. Let M be a central w-(p, q, s, ε)-molecule with $\mathfrak{N}(M) = 1$. In the proof of [8, Theorem 1], we have showed that $M = \sum_{k=0}^\infty M_k = \sum (M_k - P_k) + \sum P_k$, where

(I) each $(M_k - P_k)$ is a multiple of a w-(p, q, s)-atom centered at 0 with a sequence of coefficients in l^p,
(II) the sum $\sum P_k$ can be written as an infinite linear combination of w-(p, ∞, s)-atom centered at 0 with a sequence of coefficients in l^p.

Note that each w-(p, ∞, s)-atom centered at 0 must be an w-(p, q, s)-atom centered at 0 for any $1 < q < \infty$, and the w-(p, q, s)-atom centered at 0 coincides with the central $(n(1/p - 1/q), q, s; w, w)$-atom. We hence obtain $M = \sum_{i=0}^\infty \lambda_i a_i$, where each a_i is a central $(n(1/p-1/q), q, s; w, w)$-atom and $\sum_{i=0}^\infty |\lambda_i|^p < \infty$. It follows from Theorem C that $M \in H\dot{K}_q^{n(1/p-1/q),p}(w; w)$ and $\|M\|_{H\dot{K}_q^{n(1/p-1/q),p}(w;w)} \leqslant C = C\mathfrak{N}(M)$.

As to the central molecule of restricted type, the proof is similar and so the details are omitted. □

4. Main result and applications

The Calderón–Zygmund theorem on singular integrals, as presented by Stein [19], shows that if $k \in L^2(\mathbb{R}^n)$ satisfies

$$\int_{|x| \geqslant 2|y|} |k(x-y) - k(x)|\, dx \leqslant C \quad (\forall y \neq 0)$$

for which $\hat{k}$ is bounded, then the convolution operator $f \mapsto k * f$ is bounded on L^p, $p > 1$. We have a similar result as follows.

Theorem 2. *Let* $w \in A_1$, $0 < p \leqslant 1 < q < \infty$, *and* $\alpha = n(1/p - 1/q)$. *Assume that* $k \in L_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$ *satisfies* $\|k * f\|_{L^q_w} \leqslant C_1 \|f\|_{L^q_w}$ *and, for* $j \in \mathbb{Z}$,

$$\int_{2^{j-1} < |x| \leqslant 2^j} \frac{|k(x-y) - k(x)|^q}{|y|^{\lambda q}} w(x)\, dx \leqslant C_2 2^{-jq(\lambda + n - n/q)} w(y) \quad \text{for } 2^{j-2} \geqslant C_3 |y|, \quad (1)$$

for some $0 < \lambda \leqslant 1$ *and absolute constants* C_1, C_2, C_3. *Then there exists a constant* C *independent of* f *such that* $\|k * f\|_{\dot{K}^{\alpha,p}_q(w,w)} \leqslant C \|f\|_{H\dot{K}^{\alpha,p}_q(w,w)}$, $n/(n+\lambda) < p \leqslant 1$, *for all* $f \in \dot{K}^{\alpha,p}_q(w,w)$.

Proof. Given a central $(\alpha, q, 0; w, w)$-atom f with $\operatorname{supp} f \subseteq B_r \equiv B(0,r)$, then $\|f\|_{L^q_w} \leqslant w(B_r)^{1/q - 1/p}$ and $\int f(x)\, dx = 0$. It suffices to show $\|k * f\|_{\dot{K}^{\alpha,p}_q(w,w)} \leqslant C$, where C is independent of f. Choose $j_0 \in \mathbb{Z}$ satisfying $2^{j_0 - 3} < C_3 r \leqslant 2^{j_0 - 2}$. Write

$$\begin{aligned}
\|k * f\|^p_{\dot{K}^{\alpha,p}_q(w,w)} &= \sum_{j=-\infty}^{\infty} w(Q_j)^{\alpha p/n} \left(\int_{C_j} |k * f(x)|^q w(x)\, dx \right)^{p/q} \\
&= \left(\sum_{j=-\infty}^{j_0} + \sum_{j=j_0+1}^{\infty} \right) w(Q_j)^{\alpha p/n} \left(\int_{C_j} |k * f(x)|^q w(x)\, dx \right)^{p/q} \\
&\equiv I_1 + I_2.
\end{aligned}$$

The L^q_w boundedness of $k * f$ implies

$$I_1 \leqslant \sum_{j=-\infty}^{j_0} w(Q_j)^{\alpha p/n} \|k * f\|^p_{L^q_w} \leqslant C \|f\|^p_{L^q_w} \sum_{j=-\infty}^{j_0} w(Q_j)^{\alpha p/n}.$$

Since $w \in A_1$, we have $w \in RH_r$ for some $r > 1$. Theorem B yields

$$w(Q_j) \leqslant C w(Q_{j_0}) |Q_j|^{\delta} |Q_{j_0}|^{-\delta} \quad \text{for } \delta = (r-1)/r,$$

and we hence obtain

$$I_1 \leqslant C w(Q_{j_0})^{\alpha p/n} w(B_r)^{p/q - 1} 2^{-j_0 \alpha p \delta} \sum_{j=-\infty}^{j_0} 2^{j \alpha p \delta} \leqslant C_{\alpha,p,w} w(B_r)^{\alpha p/n + p/q - 1} \leqslant C_{\alpha,p,w}.$$

By the assumption and using Minkowski's inequality for integral, we get

$$I_2 = \sum_{j=j_0+1}^{\infty} w(Q_j)^{\alpha p/n} \left(\int_{C_j} |k * f(x)|^q w(x)\, dx \right)^{p/q}$$

$$
\begin{aligned}
&\leqslant \sum_{j=j_0+1}^{\infty} w(Q_j)^{\alpha p/n} \left\{ \int_{C_j} \left(\int_{B_r} |k(x-y)-k(x)||f(y)|\,dy \right)^q w(x)\,dx \right\}^{p/q} \\
&\leqslant Cr^{\lambda p} \sum_{j=j_0+1}^{\infty} w(Q_j)^{\alpha p/n} \left\{ \int_{C_j} \left(\int_{B_r} \frac{|k(x-y)-k(x)|}{|y|^{\lambda}} |f(y)|\,dy \right)^q w(x)\,dx \right\}^{p/q} \\
&\leqslant Cr^{\lambda p} \sum_{j=j_0+1}^{\infty} w(Q_j)^{\alpha p/n} \left\{ \int_{B_r} \left(\int_{C_j} \frac{|k(x-y)-k(x)|^q}{|y|^{\lambda q}} w(x)\,dx \right)^{1/q} |f(y)|\,dy \right\}^{p} \\
&\leqslant Cr^{\lambda p} \sum_{j=j_0+1}^{\infty} w(Q_j)^{\alpha p/n} \left\{ 2^{-j(\lambda+n-n/q)} \int_{B_r} |f(y)| w(y)^{1/q}\,dy \right\}^{p}.
\end{aligned}
$$

The Hölder's inequality gives $\int_{B_r} |f(y)||w(y)^{1/q}\,dy \leqslant \|f\|_{L^q_w} |B_r|^{1/q'}$ and Theorem B implies $w(Q_j) \leqslant C|Q_j| w(Q_{j_0}) |Q_{j_0}|^{-1}$. We thus have

$$
I_2 \leqslant Cr^{\lambda p + np/q'} 2^{-j_0 \alpha p} w(Q_{j_0})^{\alpha p/n} w(I_r)^{p/q-1} \sum_{j=j_0+1}^{\infty} 2^{j\alpha p - jp(\lambda+n-n/q)}.
$$

For $n/(n+\lambda) < p \leqslant 1$, the last summation

$$
\sum_{j=j_0+1}^{\infty} 2^{j\alpha p - jp(\lambda+n-n/q)} \leqslant C_{\alpha,p,\lambda} 2^{j_0 \alpha p - j_0 p(\lambda+n-n/q)}.
$$

Hence

$$
I_2 \leqslant C_{\alpha,p,\lambda} w(Q_{j_0})^{\alpha p/n} w(I_r)^{p/q-1} \leqslant C_{\alpha,p,\lambda,w},
$$

by which the proof is completed. □

Example 1. Let $\Omega \in L^q(S^{n-1})$ be homogeneous of degree zero, where $q > 1$ and S^{n-1} denotes the unit sphere of $\mathbb{R}^n$ $(n \geqslant 2)$ equipped with normalized Lebesgue measure $d\sigma = d\sigma(x')$. Set $k(x) = \Omega(x')/|x|^n$, with Ω satisfies the Lip_1 condition, and $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$. It is known that if $\Omega \in \mathrm{Lip}_1$ then Ω satisfies L^q-Dini condition. By L^q boundedness of $k * f$ and [9, Lemma 5], k satisfies the hypothesis (1) of Theorem 2 for $w \equiv 1$ and $\lambda = 1/q$. Using Theorem 2 for $w \equiv 1$, we get $\|k * f\|_{\dot{K}_q^{\alpha,p}(1,1)} \leqslant C\|f\|_{H\dot{K}_q^{\alpha,p}(1,1)}$, $n/(n+\lambda) < p \leqslant 1$. Note that the kernel k does not satisfy the below hypothesis (3) of Theorem 4.

Example 2. Let k satisfy the same conditions as in Example 1. For $q > 2$ and $w \equiv |x|^a$, $-1 < a \leqslant 0$. We know that $w \in A_1$. By L^q_w boundedness of $k * f$ of Theorem 4 in [9] and the [9, Lemma 5], k satisfies the hypothesis (1) of Theorem 2 for $\lambda = 1/q$. By Theorem 2, we also have $\|k * f\|_{\dot{K}_q^{\alpha,p}(w,w)} \leqslant C\|f\|_{H\dot{K}_q^{\alpha,p}(w,w)}$, $n/(n+\lambda) < p \leqslant 1$.

It is known that the Hilbert transform is not a bounded operator on $\dot{K}_q^{1-1/q,1}(\mathbb{R})$, $1 < q < \infty$ (see [12]). We shall prove that the $H\dot{K}_q^{1/p-1/q,p}(\mathbb{R})$ boundedness of Hilbert transform in Theorem 6 below, if the range of p is restricted to $0 < p \leqslant 1$. In order to show the $H\dot{K}_q^{n(1/p-1/q),p}(\mathbb{R}^n)$

(w, w) boundedness of convolution operators, we need the following estimate which can be found in [5, p. 412].

Lemma D. *Let $w \in A_q$, $q > 1$. Then, for all $r > 0$, there exists a constant C independent of r such that*

$$\int_{|x|\geqslant r} \frac{w(x)}{|x|^{nq}}\,dx \leqslant Cr^{-nq} w(I_r).$$

Theorem 3. *Let $w \in A_1$ with critical index r_w for the reverse Hölder condition. Let $0 < p \leqslant 1 < q < \infty$ and $\alpha = n(1/p - 1/q)$. Assume that $k \in L_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})$ satisfies $\|k * f\|_{L^q_w} \leqslant C_4 \|f\|_{L^q_w}$ and*

$$|k(x-y) - k(x)| \leqslant C_5 \frac{|y|^\lambda}{|x|^{n+\lambda}} \quad \text{for } |x| \geqslant C_6 |y| \tag{2}$$

for some $0 < \lambda \leqslant 1$ and absolute constants C_4, C_5, C_6. If $r_w > (n+\lambda)/\lambda$, then the operator $f \mapsto k * f$ is bounded on $H\dot{K}^{\alpha,p}_q(\mathbb{R}^n)(w, w)$, $n/(n+\lambda) < p \leqslant 1$.

Proof. Let $n/(n+\lambda) < p \leqslant 1$ and $r_w > (n+\lambda)/\lambda$. It is clear that $[n(1/p - 1)] = 0$ and $\max\{1/(r_w - 1)^{-1}, 1/p - 1\} < \lambda/n$. Choose ε satisfying $\max\{1/(r_w - 1)^{-1}, 1/p - 1\} < \varepsilon < \lambda/n$. It suffices to show that, for every central $(\alpha, q, 0; w, w)$-atom f, $k * f$ is a central w-$(p, q, 0, \varepsilon)$-molecule and its molecular norm is uniformly bounded.

For central $(\alpha, q, 0; w, w)$-atom f with $\mathrm{supp}\,(f) \subseteq B_r$, $\|f\|_{L^q_w} \leqslant w(B_r)^{1/q - 1/p}$ and $\int f(x)\,dx = 0$, let $a = 1 - 1/p + \varepsilon$, $b = 1 - 1/q + \varepsilon$. Then

$$\begin{aligned} \|k * f(\cdot) w(I_{|\cdot|})^b\|^q_{L^q_w} &= \int_{\mathbb{R}^n} |k * f(x)|^q w(I_{|x|})^{qb} w(x)\,dx \\ &= \left(\int_{|x|\leqslant C_6\sqrt{n}r} + \int_{|x|>C_6\sqrt{n}r} \right) |k * f(x)|^q w(I_{|x|})^{qb} w(x)\,dx \\ &\equiv J_1 + J_2. \end{aligned}$$

The L^q_w boundedness of $k * f$ implies

$$\begin{aligned} J_1 &\leqslant Cw(I_r)^{qb} \|k * f\|^q_{L^q_w} \\ &\leqslant Cw(I_r)^{qb} \|f\|^q_{L^q_w} \\ &\leqslant Cw(I_r)^{qb} w(I_r)^{1-q/p} \\ &\leqslant Cw(I_r)^{qa}. \end{aligned}$$

To estimate J_2 we write

$$\begin{aligned} J_2 &\equiv \int_{|x|>C_6\sqrt{n}r} |k * f(x)|^q w(I_{|x|})^{qb} w(x)\,dx \\ &= \int_{|x|>C_6\sqrt{n}r} \left| \int_{|y|\leqslant r} \{k(x-y) - k(x)\} f(y)\,dy \right|^q w(I_{|x|})^{qb} w(x)\,dx. \end{aligned}$$

Since $w \in A_1$, the Hölder inequality implies

$$\left| \int_{|y|\leqslant r} \{k(x-y) - k(x)\} f(y)\,dy \right|^q$$

$$\leqslant \left(\int_{|y|\leqslant r} |k(x-y)-k(x)|^{q'}\,dy\right)^{q/q'} \left(\int_{|y|\leqslant r} |f(y)|^q w(y)w(y)^{-1}\,dy\right)$$
$$\leqslant C\{\operatorname*{ess\,inf}_{x\in I_r} w(x)\}^{-1}\|f\|^q_{L^q_w} \left(\int_{|y|\leqslant r} |y|^{\lambda q'}|x|^{-nq'-\lambda q'}\,dy\right)^{q/q'}$$
$$\leqslant C|I_r|w(I_r)^{-1}\|f\|^q_{L^q_w} \left(\int_{|y|\leqslant r} |y|^{\lambda q'}|x|^{-nq'-\lambda q'}\,dy\right)^{q/q'}$$
$$\leqslant Cr^{n+\lambda q+nq/q'}w(I_r)^{-q/p}|x|^{-nq-\lambda q}$$

By Theorem B and Lemma D,

$$J_2 \leqslant Cr^{n+\lambda q+nq/q'}w(I_r)^{-q/p}\int_{|x|>C_6\sqrt{n}r} |x|^{-nq-\lambda q}w(I_{|x|})^{qb}w(x)\,dx$$
$$\leqslant Cr^{n-nqb+\lambda q+nq/q'}w(I_r)^{qb-q/p}\int_{|x|>C_6\sqrt{n}r} |x|^{nqb-nq-\lambda q}w(x)\,dx$$
$$\leqslant Cw(I_r)^{qa}.$$

Hence $\|k*f(\cdot)w(I_{|\cdot|})^b\|_{L^q_w}\leqslant Cw(I_r)^a$ and

$$\mathfrak{N}_w(k*f)=\|k*f\|^{a/b}_{L^q_w}\|k*f(\cdot)w(I_{|\cdot|})^b|^{1-a/b}_{L^q_w}\leqslant Cw(I_r)^{(1/q-1/p)a/b}w(I_r)^{a(1-a/b)}=C.$$

To show the vanishing moment conditions of $k*f$ for central $(\alpha,q,s-1)$-atom f with respect to (w,w), we first claim the case $w\equiv 1$. Since $\|k*f(x)|x|^{nb}\|_{L^q}\leqslant Cr^{na}<\infty$, Hölder's inequality implies

$$\int_{|x|>1}|k*f(x)|\,dx\leqslant\|k*f(x)\cdot|x|^{nb}\|_q\left(\int_{|x|>1}|x|^{-q'nb}\,dx\right)^{1/q'}<\infty$$

and

$$\int_{|x|\leqslant 1}|k*f(x)|\cdot|x|^{|\alpha|}\,dx\leqslant C\|k*f\|_q<\infty.$$

Let $\hat{f}$ be the Fourier transform of f. The moment condition of f gives $\hat{f}(0)=0$ which implies $\widehat{k*f}(0)=\hat{k}(0)\hat{f}(0)=0$ and hence $\int k*f(x)\,dx=0$. For general $w\in A_1$, let f be a central $(\alpha,q,s-1;w,w)$-atom and $f_1=w(B_r)^{1/p}|B_r|^{-1/p}f$. Then f_1 is a multiple of a central $(\alpha,q,s-1;1,1)$-atom. We have

$$\int_{\mathbb{R}^n}k*f(x)\,dx=C\int_{\mathbb{R}^n}k*f_1(x)\,dx=0,$$

by which the theorem is proved. □

Remark 1. We note that the condition (2) implies condition (1) provided $w\in A_1$. To see this, for $f\in L_{\mathrm{loc}}(\mathbb{R}^n)$, let Mf denoted the Hardy–Littlewood maximal function of f defined by

$$Mf(y)=\sup_{r>0}\frac{1}{|B(y,r)|}\int_{B(y,r)}|f(x)|\,dx.$$

If k satisfies (2) for some $0<\lambda\leqslant 1$ and absolute constants C_5, C_6, then we choose $C_3 = \max\{1, C_6/2\}$. For $2^{j-2}\geqslant C_3|y|$ and $|x| > 2^{j-1}$, we have $|x| > 2^{j-1}\geqslant 2C_3|y| = \max\{2, C_6\}\cdot|y|$ and

$$\begin{aligned}\int_{2^{j-1}<|x|\leqslant 2^j} \frac{|k(x-y)-k(x)|^q}{|y|^{\lambda q}} w(x)\,dx &\leqslant C_5 \int_{2^{j-1}<|x|\leqslant 2^j} x^{-nq-\lambda q} w(x)\,dx \\ &\leqslant C2^{-jq(n+\lambda)} \int_{|x-y|\leqslant c2^j} w(x)\,dx \\ &\leqslant C2^{-jq(n+\lambda-n/q)} Mw(y) \\ &\leqslant C2^{-jq(n+\lambda-n/q)} w(y),\end{aligned}$$

since $|x| \approx |x-y|$ for $|x| > 2|y|$ and $w\in A_1$ implies $Mw(y)\leqslant Cw(y)$.

If we use a stronger condition than (2), then the weighted L^q-boundedness of the convolution operator $k * f$ can be replaced by the L^∞-boundedness of $\hat{k}$.

Theorem 4. *Let $w\in A_1$ with critical index r_w for the reverse Hölder condition, $\alpha = n(1/p - 1/q)$, $1<q<\infty$ and $n/(n+s) < p\leqslant n/(n+s-1)$ where $s\in\mathbb{N}$. Assume that $k\in C^\infty(\mathbb{R}^n\backslash\{0\})$ with $|\hat{k}|\leqslant C_7$ satisfies*

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha} k(x)\right|\leqslant C_8|x|^{-n-|\alpha|} \quad \text{for all multi-indices } |\alpha|\leqslant s, \tag{3}$$

where C_7 and C_8 are absolute constants. If $r_w > n+s$ then the operator $Tf := k*f$ is bounded on $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)(w,w)$.

Remark 2. Theorem 4 extends Lu and Yang's result [15, Theorem 4.3] to the weighted case.

Remark 3. We note that condition (3) implies condition (2) when $C_6 > 1$. In fact, the mean value theorem implies

$$|k(x-y)-k(x)|\leqslant|\nabla k(x-ty)||y|\leqslant C_8|x-ty|^{-n-1}|y| \quad \text{for some } t\in[0,1].$$

For $|x|\geqslant C_6|y|$, we have $|x-ty|\geqslant|x|-|ty|\geqslant c|x|$ and hence

$$|k(x-y)-k(x)|\leqslant C|x|^{-n-1}|y|.$$

Proof of Theorem 4. Since $n/(n+s) < p\leqslant n/(n+s-1)$ and $r_w > n+s$, we have $s-1 = [n(1/p-1)]$ and so a number ε can be chosen satisfying $\max\{(s-1)r_w(r_w-1)^{-1}n^{-1} + (r_w-1)^{-1}, 1/p-1\} < \varepsilon < s/n$. It suffices to prove that, for every central $(\alpha, q, s-1; w, w)$-atom f, Tf are w-$(p, q, s-1, \varepsilon)$-molecules and $\mathfrak{N}_w(Tf)\leqslant C$ with C independent of f.

Given central $(\alpha, q, s-1; w, w)$-atom f with $\operatorname{supp}(f)\subseteq B_R$, we have $\|f\|_{L^q_w}\leqslant w(B_R)^{1/q-1/p}$ and $\int f(x)x^\alpha\,dx = 0$ for $0\leqslant|\alpha|\leqslant s-1$. Let $a = 1-1/p+\varepsilon$ and $b = 1-1/q+\varepsilon$. Then

$$\begin{aligned}\|Tf(x)w(I_{|x|})^b\|^q_{L^q_w} &= \int_{\mathbb{R}^n} |Tf(x)|^q w(I_{|x|})^{qb} w(x)\,dx \\ &= \left(\int_{|x|<2\sqrt{n}R} + \int_{|x|\geqslant 2\sqrt{n}R}\right)|Tf(x)|^q w(I_{|x|})^{qb} w(x)\,dx \\ &:= I_1 + I_2.\end{aligned}$$

It follows from [5, p. 411, Theorem 3.1] to get the L^q_w boundedness of T, which combines with Theorem A yields

$$I_1 \leqslant C_w w(I_R)^{qb} \|f\|^q_{L^q_w} \leqslant C_w w(I_R)^{qa}.$$

To estimate I_2 we write

$$\begin{aligned} I_2 &:= \int_{|x|\geqslant 2\sqrt{n}R} |k * f(x)|^q w(I_{|x|})^{qb} w(x)\,dx \\ &= \int_{|x|\geqslant 2\sqrt{n}R} \left| \int_{I_R} \left\{ k(x-y) - \sum_{|\alpha|=0}^{s-1} \frac{1}{\alpha!} D^\alpha k(x)(-y)^\alpha \right\} f(y)\,dy \right|^q w(I_{|x|})^{qb} w(x)\,dx. \end{aligned}$$

Taylor's theorem and $w \in A_1$ give

$$\begin{aligned} &\left| \int_{I_R} \left\{ k(x-y) - \sum_{|\alpha|=0}^{s-1} \frac{1}{\alpha!} D^\alpha k(x)(-y)^\alpha \right\} f(y)\,dy \right|^q \\ &\quad \leqslant C_n R^{qs} |x|^{-nq-qs} \left| \int_{B_R} f(y)\,dy \right|^q \\ &\quad \leqslant C_n R^{qs} |x|^{-nq-qs} \left(\operatorname*{ess\,inf}_{x\in I_R} w(x) \right)^{-q} w(B_R)^{q-1} \|f\|^q_{L^q_w} \\ &\quad \leqslant C_n R^{nq+qs} |x|^{-nq-qs} w(B_R)^{-q/p} \quad \text{for } |x| \geqslant 2\sqrt{n}R. \end{aligned}$$

Thus, Theorem B and Lemma D imply

$$\begin{aligned} I_2 &\leqslant C R^{nq+qs} w(B_R)^{-q/p} \int_{|x|\geqslant 2\sqrt{n}R} |x|^{-nq-qs} w(I_{|x|})^{qb} w(x)\,dx \\ &\leqslant C_w R^{nq+qs-nqb} w(B_R)^{-q/p+qb} \int_{|x|\geqslant 2\sqrt{n}R} |x|^{nqb-nq-qs} w(x)\,dx \\ &\leqslant C_w w(I_R)^{qa}. \end{aligned}$$

Hence

$$\|Tf(x) w(I_{|x|})^b\|_{L^q_w} \leqslant C_w w(I_R)^a \tag{4}$$

and

$$\mathfrak{N}_w(Tf) = \|Tf\|^{a/b}_{L^q_w} \|Tf(x) w(I_{|x|})^b\|^{1-a/b}_{L^q_w} \leqslant C_w.$$

Now, we are going to show the vanishing moment conditions of Tf for central $(\alpha, q, s-1)$-atom f with respect to $(1, 1)$. Plugging in $w \equiv 1$ in (4), we have $\|Tf(x)|x|^{nb}\|_q \leqslant CR^{na} < \infty$ whenever f is a central $(\alpha, q, s-1; 1, 1)$-atom. We first claim $Tf(x) \cdot x^\alpha \in L^1$ for $|\alpha| \leqslant s-1$. Since $Tf(x) \cdot |x|^{nb} \in L^q$, we use Hölder's inequality to get

$$\int_{|x|>1} |Tf(x)| \cdot |x|^{|\alpha|}\,dx \leqslant \|Tf(x) \cdot |x|^{nb}\|_q \left(\int_{|x|>1} |x|^{q'|\alpha| - q'nb}\,dx \right)^{1/q'} < \infty$$

and

$$\int_{|x|\leqslant 1} |Tf(x)| \cdot |x|^{|\alpha|}\,dx \leqslant \|Tf\|_q \left(\int_{|x|\leqslant 1} |x|^{q'|\alpha|}\,dx \right)^{1/q'} < \infty.$$

It follows from [20, Lemma 9.1] that $\hat{f}$ is $(s-1)$th order differentiable and $\hat{f}(\xi) = O(|\xi|^s)$ as $|\xi| \to 0$. If we set e_j to be the jth standard basis of $\mathbb{R}^n$ and $\Delta^\alpha_{|h|} = \Delta^{\alpha_1}_{|h|e_1}\Delta^{\alpha_2}_{|h|e_2}\cdots\Delta^{\alpha_n}_{|h|e_n}$, then the boundedness of $\hat{k}$ implies

$$\begin{aligned}\left|\int_{\mathbb{R}^n} Tf(x)\cdot x^\alpha\,dx\right| &= C_n\left|D^\alpha(\widehat{Tf})(0)\right| \\ &= C_n\left|\lim_{|h|\to 0}|h|^{-|\alpha|}\Delta^\alpha_{|h|}(\hat{k}\hat{f})(0)\right| \\ &\leqslant C_n\lim_{|h|\to 0}|h|^{s-|\alpha|} \\ &= 0 \quad \text{for } |\alpha|\leqslant s-1.\end{aligned}$$

We hence prove that, for central $(\alpha, q, s-1; 1, 1)$-atom f,

$$\int_{\mathbb{R}^n} Tf(x)\cdot x^\alpha\,dx = 0 \quad \text{for } |\alpha|\leqslant s-1. \tag{5}$$

For general case $w \in A_1$, let f be a central $(\alpha, q, s-1; w, w)$-atom and let $f_1 = w(B_R)^{1/p}|B_R|^{-1/p}f$. Then f_1 is a multiple of a central $(\alpha, q, s-1; 1, 1)$-atom. By (5),

$$\int_{\mathbb{R}^n} Tf(x)x^\alpha\,dx = C\int_{\mathbb{R}^n} Tf_1(x)x^\alpha\,dx = 0 \quad \text{for } |\alpha|\leqslant s-1.$$

Thus, the theorem is proved. □

If the regularity on the kernel is strengthened, the condition $r_w > n+s$ can be drop off and the range of p can be extended to $(0, 1]$.

Corollary 5. *Let $w \in A_1$ and $0 < p \leqslant 1 < q < \infty$. Assume that $k \in C^\infty(\mathbb{R}^n\backslash\{0\})$ with $|\hat{k}|\leqslant C_9$ satisfies*

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha k(x)\right|\leqslant C_{10}|x|^{-n-|\alpha|} \quad \textit{for all multi-indices } \alpha, \tag{6}$$

where C_9 and C_{10} are absolute constants. Then the operator $Tf := k * f$ is bounded on $H\dot{K}_q^{n(1/p-1/q),p}(\mathbb{R}^n)(w, w)$.

Proof. Given $p \in (0, 1]$, we choose $s \in \mathbb{N}$ such that $n/(n+s) < p \leqslant n/(n+s-1)$. We then choose a number ε satisfying $\max\{(s-1)r_w(r_w-1)^{-1}n^{-1} + (r_w-1)^{-1}, 1/p-1\} < \varepsilon < ts/n$, for $t \in \mathbb{N}$ and $t > ((s-1)r_w + n)/s(r_w-1)$. It suffices to prove that, for every central $(n(1/p-1/q), q, ts-1; w, w)$-atom f, Tf are w-$(p, q, s-1, \varepsilon)$-molecules and $\mathfrak{N}_w(Tf)\leqslant C$ with C independent of f. The corollary follows from the same arguments as in the proof of Theorem 4. □

Remark 4. There are results similar to Theorems 2–4 and Corollary 5 for the spaces $HK_q^{\alpha,p}(\mathbb{R}^n)(w, w)$. We leave details to readers.

It was shown in [11] that the Hilbert transform and Riesz transforms are bounded on H^p_w for $w \in A_1$ and $0 < p \leqslant 1$. Recall that on $\mathbb{R}^1$, the Hilbert transform $f \mapsto Hf$ is a convolution operator with kernel $k(x) = 1/(\pi x)$. On $\mathbb{R}^n, n \geqslant 2$, let $R_j,\ j = 1, 2, \ldots, n$, denote the Riesz transforms defined by

$$R_j f = \text{p.v. } k_j * f(x) \quad \text{where } k_j(x) = \pi^{-(n+1)/2}\Gamma\Big(\frac{n+1}{2}\Big)\frac{x_j}{|x|^{n+1}}.$$

It is easy to check that the kernels k and k_j satisfy condition (6) and so we have the following theorem.

Theorem 6. *Let $w \in A_1$ and $0 < p \leqslant 1 < q < \infty$. The Riesz transforms are bounded on $H\dot{K}_q^{n(1/p-1/q),p}(\mathbb{R}^n)(w, w)$ and $HK_q^{n(1/p-1/q),p}(\mathbb{R}^n)(w, w)$. For $n = 1$, the Hilbert transform is bounded on $H\dot{K}_q^{1/p-1/q,p}(\mathbb{R})(w, w)$ and $HK_q^{1/p-1/q,p}(\mathbb{R})(w, w)$.*

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